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THE APPLICATION OF A CLASS OF INFINITE-DIMENSIONAL LIE GROUPS TO THE DYNAMICS OF AN INCOMPRESSIBLE FLUID†

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A group approach to the analysis of the Euler and Navier–Stokes equations is proposed. Generalized groups of currents, i.e. groups of linear transformations of the tangent spaces of a smooth manifold that conserve a given structure on it and depend smoothly on the point, are considered. Semidirect products of such groups and groups that act on the manifold and conserve the structure are constructed. It is shown that the infinite-dimensional group obtained is a Lie–Frechet group of the second kind. Using the construction obtained a group interpretation of the solutions of the travelling wave type in multidimensional hydrodynamics is given. A general description of the solutions that are constructed of the stationary flow and the vector field of the compact group of diffeomorphisms of the flow domain that conserve a volume, is presented. The group approach, which is based on the constructions of groups of currents and its generalizations, enables one to describe in a unified manner various physical phenomena: the non-linear dynamics of the magnetization of ferromagnetic materials, some classes of flows of an incompressible fluid and certain objects of quantum cosmology. © 2003 Elsevier Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

Examples of group analysis in mechanics were given in [1] for Hamiltonian systems by using the apparatus of local Lie groups and also for a certain class of flows of an incompressible fluid [2] by the application of finite-dimensional Lie groups. It was stated [3] that in physical applications one should use infinite-dimensional Lie groups.

In this paper the problem of the non-local solvability of multidimensional equations for an incompressible ideal or viscous fluid is studied. The domain of fluid flows is some manifold M provided with a Riemannian metric. The fluid velocity fields form a space SVect(M) of divergence-free vector fields on M. Provided with a Poisson bracket (a commutator) of vector fields $[u, \upsilon]$ the space SVect(M) becomes a Lie algebra. Its Lie group is the infinite-dimensional group SDiff(M) of diffeomorphisms that conserve a volume element of the manifold M, which can be treated as the configuration space of fluxes of incompressible fluid in M [4, 5]. The transport of fluid particles (an advection) [6] is given by the action of the curve $g(t)$ from the group SDiff(M) on M. The fluid velocity field $u(t)$ is obtained from the tangent vector to the curve $Tg(t)$ by applying the right-hand shift by the element $g(t)^{-1}$ from $g(t)$ into the unit of the group SDiff(M). This gives a relation between the Eulerian and Lagrangian portrait of incompressible fluid flows. The curve $u(t)$ in the Lie algebra SVect(M) obeys Euler's equations (an ideal fluid) or the Navier–Stokes equations (a viscous fluid). In the general case the problem of constructing the curve $g(t)$ in the Lie algebra SVect(M) with given initial vector field $u = u(0)$ that obeys the Euler of Navier–Stokes equations and is extendable to infinity in time has not been solved for multi-dimensional (the dimensionality is three or more) hydrodynamics.

Below we construct infinite-dimensional subalgebras in the Lie algebra of divergence-free vector fields $g \subset$ SVect(M), which possess the property that for the initial conditions $u(0)$ specified by vector fields from these subalgebras ($u(0) \in g$), the solutions of the Euler and Navier–Stokes equations $u(t)$ remain of the same variety ($u(t) \in g$) and can be continued to infinity in time.

The infinite-dimensional Lie groups of currents $G(M, K)$, namely, groups that are formed by smooth pointwise mappings from the manifold M to the given finite-dimensional Lie group K are used. For example, the group of currents $G(M, SO(n))$ (where $SO(n)$ is a proper orthogonal group) is a configuration space in the problem of the non-linear dynamics of the magnetization of ferromagnetic materials (a generalized solid body) described by the Landau–Lifshits equation [7]. In the context of

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this physical formulation of the problem a generalization of the group of currents $G(M, SO(n))$ was proposed [8], namely, the group $O(M)$ of pointwise orthogonal transformations of tangent spaces on the compact oriented Riemannian manifold M. Its Lie algebra is the Lie algebra $o(m)$ of pointwise skew-symmetric transformations of tangent spaces on M. Generalized groups of currents, connected with other geometric structures, for example, conformal structures can also be considered.

It the general case the geometric structure desired is given by a certain G-structure [9, p. 332], i.e. by the bundle $P \to M$ with the structure group $G \subset GL(V)$, where $V = \mathbf{R}^n$ and $n = \dim M$, that is, at some point the space V is isomorphic to a space tangent to M. As a result, a G-structure of general form is obtained by reduction of the principal bundle $F(M) \to M$ of the frames on M from the structure group $GL(V)$ (complete linear) to some subgroup G of it. For a given G-structure P on M one can construct a generalized group of currents R, which acts in the tangent bundle TM by layerwise transformations $G(TM)$ T_xM that depend smoothly on the base point x and conserve the geometric structure induced by the G-structure P in the tangent space T_xM. Thus, elements of the group R transform the frame set of the G-structure into itself. They can be represented as smooth sections of the bundle $G(M) \to M$ associated with the principal bundle determined by the G-structure. The Lie algebra r of the group R consists of smooth sections of the vector bundle $g(M) \to M$. Here the Lie algebras $g(T_xM)$ of the groups $G(T_xM)$ that in the tangent spaces T_xM conserve the geometric structure determined by the G-structure are layers. The layers are isomorphic to the Lie algebra g of the Lie group G.

In this paper, groups, which are a semidirect product of generalized groups of currents associated with geometric structures and finite-dimensional Lie groups consisting of isomorphisms of these structures, are constructed. The infinite-dimensional Lie groups obtained in this way are embedded into the group SDiff(M). A group analysis of the classes of incompressible fluid flows obtained is carried out, and the corresponding solutions of the Euler and Navier–Stokes equations are obtained from this analysis.

2. THE NECESSARY APPARATUS FROM THE THEORY OF INFINITE-DIMENSIONAL LIE GROUPS

We will dwell briefly on the structural properties of infinite-dimensional Lie groups. The group of diffeomorphisms SDiff(M) has a structure of the Lie–Frechet group [10, 11], whereas the group of currents $G(M, K)$ also possesses the stronger structure of the Campbell–Hausdorff group, or a Lie group of the first kind. A group of the first kind is such a Lie–Frechet group that possesses canonical coordinates of the first kind, that is, its Lie exponential defines a local map in the unit element of the group [12, 13]. The generalized group of currents is also a Lie group of the first kind [8].

In [13] the concept of an infinite-dimensional Lie group of the second kind was introduced as possessing local coordinates that are an analog of canonical coordinates of the second kind of finite-dimensional Lie groups. Consequently, the group of the second kind is such a Lie–Frechet group G that its Lie algebra g can be expanded in a direct sum of topological vector spaces $g = V_1 + \dots + V_k$ and a product of k Lie exponentials, i.e. the map $(\upsilon_1, \dots, \upsilon_k) \to \exp(\upsilon_1) \dots \exp(\upsilon_k)$, specifies a local map in the unit of a group. For this class of groups a Lie theory was constructed, which is similar to the finite-dimensional theory (three Lie theorems were proved), which could not be done for all group SDiff(M).

The following construction will be used many times.

Theorem 2.1. If the finite-dimensional Lie group K acts on the manifold M by automorphisms of the G-structure P, then one can construct the semidirect product $B = KR$ of the group K and the generalized group of currents R of the G-structure pointwise automorphisms, which is a Lie–Frechet group of the second kind and ILH is the Lie group.

Proof. Let the action of the group K on the group R be given. In what follows it will be convenient to denote the tangent vector $a \in T_pM$ by the pair (p, a), that is, $\pi(a) = p$, where π: $TM \to M$ is a projection of the tangent bundle. For the element $k \in R$ we put $Tk(p, a) = (k(p), dk|_p(a))$. Then for the element $o \in R$ we defined

$$k(o) = (Tk)o(Tk^{-1}) \tag{2.1}$$

From the condition of the theorem we have that $k(o) \in R$ and K acts on R by automorphisms.

We can now construct the semidirect product B of the groups K and R. In B we introduce the operation

$$(l, p)(k, o) = (lk, pl(o)) \tag{2.2}$$

It is well known that B with operation (2.2) is a group. The pair (Id, Id) is the unit element. The inverse element has the form $(k, o)^{-1} = (k^{-1}, k^{-1}(o^{-1}))$.

Let

$$(k, o)(p, a) = (k(p), o|_{k(p)}(dk|_p(a))) \tag{2.3}$$

It can be directly verified that the mapping (2.3) specifies the action of the group B on the tangent bundle TM.

Consider the Lie algebra r. Let k be a Lie algebra of the group K; this algebra consists of the Killing (conserving the G-structure) vector fields on M. We introduce the mapping F: $k \times r \to B$. We put

$$F(q, f) = \exp(q)\exp(f), \quad q \in \mathrm{k}, \quad f \in \mathrm{r}$$

Let $U \subset k$ be a neighbourhood of the injectivity of the exponential mapping of the group K, and $V \subset r$ be a neighbourhood of zero element consisting of sections of the bundle $g(M) \to M$ that pointwisely belong to the domains of injectivity of the exponential mapping in a layer (exp: $g(T_xM) \to G(T_xM)$). Then the restriction of the mapping F to $U \times V$ defines canonical coordinates of the second kind on B and specifies a local map in the group unit. At an arbitrary point $p = (g, s) \in B$ the local map $A(g, s)$ can be given by the mapping

$$F(q, f) = g\exp(q)s\exp(f), \quad (q, f) \in U \times V$$

A transition from the map $A(g, s)$ to the map $A(j, r)$ leads to the equations

$$h = g\exp(q)s\exp(f) = j\exp(p)r\exp(k), \quad h \in A(g, s) \cap A(j, r)$$

From the uniqueness of the element expansion in the semidirect product of groups we obtain $g\exp(q) = j\exp(p)$, $s\exp(f) = r\exp(k)$. From this we conclude that the transition functions on the group B are reduced to a pair of transition functions of canonical coordinates for the finite-dimensional Lie group K and the generalized group of currents R. As a result, the atlas $\{A(g, s)\}$ constructed defines the structure of the Lie–Frechet group on B. From (2.2) the operation of multiplying m by B can be represented in the form of the superposition $m = (m_K, m_R(\mathrm{Id}, Aut(K)))$ of multiplication in K, multiplication in R and the action of K on R by the automorphisms according to formula (2.1). Hence follows the smoothness of group operations in the atlas constructed.

To introduce the structure of the Lie group ILH we imbed the group R into the group R^n, which consists of the bundle sections $C(M) \to M$ of Sobolev's class W^n; its Lie algebra will be the space r^n consisting of sections of the bundle $g(M) \to M$ of class W^n. The group operation in the generalized group of currents R is smooth in the class W^n since this is reduced to pairwise multiplications of coordinate functions of sections in the atlas constructed. The action of the Lie algebra k is reduced to multiplications of given smooth functions by the first derivatives of the coordinate functions of sections from r^n, and hence this operation lowers by one the Sobolev smoothness of these sections, that is, it transforms the Sobolev class W^n into W^{n-1}. Hence, using the atlas constructed, one can show that the action K on R by automorphisms are smooth in the ILH-sence, i.e. it has the class W^m because it is a mapping from W^{n+m+1} into W^n. The theorem is proved.

As a compact manifold we next consider the spherical bundle $S(M) \to M$ over the n-dimensional Riemannian manifold M with a layer being an $(n-1)$-dimensional unit sphere S^{n-1}.

Proposition 2.1. Under the conditions of Theorem 2.1 let the Lie group K be transitive on the manifold M, and let the groups of automorphisms G_x in layers be transitive on rays in T_xM, and moreover, for any element $g \in G_x$ the element $g((\det g)^{-1}\mathrm{Id}) \in G$.

Then the natural action of the group B on the tangent bundle TM induces a transitive action on the spherical bundle $S(M)$ in the case of the Riemannian manifold M. The subgroup $N = \{\mathrm{Id}, \lambda(x)\mathrm{Id}_X\}$, where $\lambda(x)\mathrm{Id}_X$ is a pointwise homothety of the linear spaces T_xM, is a normal subgroup of non-efficiency of this action, and the action of the factor-group $Q = B/N$ on $S(M)$ is an efficient one.

Proof. In the case of action on the manifold with Riemannian metric, the action on $S(M)$ is defined by the action on TM according to the formula

$$(k, o)(p, a) = (k(p), o(dk(a))/\det(o(dk(a))))$$

and since the point (p, a) belongs to a spherical bundle, we have $|a| = 1$.

The subgroup N remains the points $S(M)$ to be fixed. In the arbitrary conjugacy class $(u, f)N$ lies a single element of the form (u, ϕ) such that ϕ is a pointwise unimodal transformation, namely, this is the element $\phi = f/\det f$. From this it follows that the factor-group $Q = B/N$ acts efficiently on $S(M)$.

Example 2.1. Let $M = T^2$ be a two-dimensional torus with a standard metric, and $K = T^2$. In this case $S(M) \cong T^3$. In the standard coordinates (x, y, z) on T^3, taken modulo 2π, vector fields of the form

$$u = a\partial x + b\partial y + f(x, y)\partial z, \quad , \quad a, b \in \mathbf{R}$$

make up the Lie algebra b of the group B.

Example 2.2. Take $M = S^n$ (an n-dimensional sphere) with a conformal structure. As K we take the group $SO(1, n + 1)$ of conformal transformations S^n (when $n = 2$ this is the Lorentz group).

3. APPLICATIONS TO THE DYNAMICS OF AN IDEAL INCOMPRESSIBLE FLUID

In what follows, the case when K is a group of isometry of a Riemannian metric on the manifold M, and R is a group $O(M)$ of pointwise orthogonal transformations, will be important. In this case we will consider as a domain of fluid flows the manifold $S(M)$, i.e. the spherical bundle $S(M) \to M$. The manifold $S(M)$ has a dimension that is one greater than M; as an example, if M is a surface, $S(M)$ is a three-dimensional manifold. $S(M)$ is a submanifold of the tangent bundle TM, which is invariant under the natural action of groups K and R on TM. The group $B = KR$ is imbedded into the group SDiff(SM)) of diffeomorphisms that conserve the volume element $S(M)$, and, therefore, into the configuration space of incompressible fluid in $S(M)$.

We will establish the relation of this construction with Euler's equations for an ideal incompressible fluid

$$\partial \upsilon/\partial t + \nabla_\upsilon \upsilon = \nabla p \tag{3.1}$$

It will be also convenient to use a representation of Euler's equations in the form

$$\partial \upsilon/\partial t = -\mathrm{ad}\,\upsilon^*(\upsilon) \tag{3.2}$$

Let N be a Riemannian manifold, which is the domain of fluid flows. Then $\mathrm{ad}\,\upsilon^*$ is an operator of co-adjoint representation in the Lie algebra SVect(N) of divergence-free vector fields on N, where the conjugacy operation corresponds to scalar product in the space of divergence-free vector fields [4]

$$\langle u, \upsilon\rangle = \int_M \langle u(x), \upsilon(x)\rangle dx \tag{3.3}$$

that is $\langle \mathrm{ad}\,\upsilon^*(u), w\rangle = \langle u, \mathrm{ad}\,\upsilon(w)\rangle = \langle u, [\upsilon, w]\rangle$.

The Riemannian metric (3.3) is defined in SVect(N), which is a space tangential to the unit of the group of diffeomorphisms SDiff(N) that conserve the volume element N. This metric is continued to the right-invariant metric, which is the kinetic energy on the group SDiff(N) [4].

Below we analyse Example 2.1 in these terms. Let b be a Lie algebra of the group B. We will study the velocity fields of the Lie algebra B as vector fields of an ideal incompressible fluid on T^3. For the vector field $u = (a, b, f)$ on T^3 Euler's equations lead to the system

$$\partial a/\partial t = 0, \quad \partial b/\partial t = 0, \quad \partial f/\partial t + a\partial f/\partial x + b\partial f/\partial y = 0$$

From this one can obtain a solution of Euler's equations

$$u^t = (a, b, f(x - at, y - bt))$$

A generalization of the subgroup B, which also leads to solutions of three-dimensional hydrodynamics that are extendable to infinity in time, can be constructed. We consider vector fields on T^3 of the form $w = \upsilon + f(x, y)\partial z$, where $\upsilon = a(x, y)\partial x + b(x, y)\partial y$, that is, a and b are no longer constants, but are functions on T^2, and υ is a divergence-free field on T^2. It is obvious that such vector fields make up a Lie algebra; we will denote this algebra by s and the corresponding group of diffeomorphisms by S. For the vector field w Euler's equations lead to the system

$$\partial\upsilon/\partial t + \langle\nabla\upsilon, \upsilon\rangle = \nabla p, \quad \partial f/\partial t + a\partial f/\partial x + b\partial f/\partial y = 0$$

The equations of the vector field υ are Euler's equations on T^2. In two-dimensional hydrodynamics Euler's equations have solutions that are extendable to infinity in time. We will denote this solution by υ^t, and the corresponding flow of an ideal incompressible fluid on T^2 by g^t. Then it can be directly verified that the vector field $w^t = (\upsilon^t, f(g^{-t}(x, y))$ will be a solution of the original Euler's equations on T^3 with the initial condition $w^0 = (\upsilon^0, f(x, y))$.

Proposition 3.1. The solutions of Euler's equations with initial conditions $u = u^0 \in s$ give the curves $u^t \in s$ extendable to infinity in time. The corresponding flows U^t of an ideal incompressible fluid lie in the group S.

We will now formulate a general statement, which gives integrable solutions of Euler's equations in multidimensional hydrodynamics. We recall that the solution of Euler's equations $\upsilon^t = \upsilon$ is referred to as a stationary vector field if it is constant in time [14, p. 69]. We will first prove the following proposition.

Proposition 3.2. Suppose that, on the oriented compact Riemannian manifold M, the vector field υ from the Lie algebra k of the compact Lie group K of the Riemannian metric automorphisms is given. Then the vector field u is stationary.

Proof. We will use form (3.2) of Euler's equations. Since u is the vector field of a compact group that conserves the Riemannian metric on M, the operator $\mathrm{ad}\,u$ is skew-symmetric in the space SVect(M) of divergence-free vector fields on M with a scalar product that is the kinetic energy. If we denote the Lie algebra of the group K by k and the orthogonal complement to k by P, it can easily be shown that $\mathrm{ad}\,u(P) \subset P$. It is obvious that $\mathrm{ad}\,u(k) \subset k$. Then $(\mathrm{ad}\,u)^*(k) \subset k$ also. We will show that $(\mathrm{ad}\,u)^*(u) = 0$. We take $\upsilon \in k$ and we will then have $\langle(\mathrm{ad}\,u)^*(u), \upsilon\rangle = \langle u, \mathrm{ad}\,u(\upsilon)\rangle = -\langle u, \mathrm{ad}\,\upsilon(u)\rangle$.

However, since $\upsilon \in k$, the operator $\mathrm{ad}\upsilon$ is skew-symmetric, and from this it follows that $\langle u, \mathrm{ad}\upsilon(u)\rangle = 0$. But since $(\mathrm{ad}\,u)^*(u) \subset k$, then $(\mathrm{ad}\,u)^*(u) = 0$, and the vector field u is stationary.

For later use we need a standard action of the diffeomorphism g on the vector field υ or the function f, which we will denote by an asterisk subscript:

$$g_*\upsilon(x) = dg|_{g^{-1}x}\upsilon|_{g^{-1}x}, \quad g_*f(x) = f(g^{-1}x)$$

Theorem 3.1. Suppose that, on the compact oriented Riemannian manifold M, there are two divergence-free vector fields u, υ that obey the following conditions

(1) u is a vector field from the Lie algebra for the compact. Lie group of auto-morphisms of the Riemannian metric on M;

(2) the vector field υ is stationary;

(3) $\nabla_u\upsilon = 0$ or conditions $3'\nabla_\upsilon u = 0$.

Then the vector field

$$w^t = u + g_*^{-t}(t) \text{ under condition 3 or } u + g_*^t(\upsilon) \text{ under condition } 3'$$

will be a solution of Euler's equations for an ideal incompressible fluid with the initial condition $u + \upsilon$.

Here g^t is the flux of the vector field u.

Proof. From Proposition 3.2 it follows that the vector field u is stationary, and therefore, $\nabla_u u = \nabla q$ (a gradient vector field). From the stationarity of the vector field υ it follows that $\nabla_\upsilon\upsilon = \nabla f$. Using condition 3, we calculate

$$\partial t(w^t)_{t=\tau} = \partial t g_*^{-t}(\upsilon)_{t=\tau} = [u, g_*^{-\tau}(\tau)] = \nabla_u g_*^{-\tau}(\upsilon) - \nabla_* u \quad (\nabla_* = \nabla_{g_*^{-\tau}(\upsilon)})$$

Note that $g_*(u) = u$

If now we lift the vector field u and the action of the diffeomorphism g_* on the tangent bundle TM, there a similar identity will be conserved. Since g_* conserves the Riemannian structure, we have $\nabla_u g_*^{-\tau}(\upsilon) = g_*^{-\tau}\nabla_u(\upsilon) = 0$ according to condition 3.

Let us calculate $\nabla_{w^t} w^t$. From the conditions of the theorem and the preceding calculations we have

$$\nabla_{w^t} w^t = \nabla_u u + \nabla_u g_*^{-\tau}(\upsilon) + \nabla_* u + \nabla_* g_*^{-\tau}(\upsilon) = \nabla_u u + \nabla_* u + \nabla_* g_*^{-\tau}(\upsilon)$$

Since $g^{-\tau}$ conserves the metric tensor, then

$$\nabla_* g_*^{-\tau}(\upsilon) = g_*^{-\tau}\nabla_\upsilon \upsilon = g_*^{-\tau}\nabla f = \nabla g_*^{-\tau} f$$

Hence, we have

$$\partial t(w^t) = \nabla_{w^t} w^t = \nabla g_*^{-\tau} f + \nabla_u u = \nabla g_*^{-\tau} f + \nabla q$$

From this it follows that w^t satisfied Euler's equations. The theorem with condition 3 is proved. A version with condition 3′ can be proved similarly.

Example 3.1. Consider vector fields on T^3, $\upsilon = g + h$, where $q = a\partial x + b\partial y + c\partial z$, h is a stationary divergence-free vector field, and a, b and c are constants. We have $\nabla_h q = 0$, i.e. condition 3′ of Theorem 3.1 is satisfied, and we obtain that a solution of Euler's equations with initial data υ has the form

$$\upsilon^t = (a, b, c) + h(x - at, y - bt, z - ct)$$

In particular, let a, b, c, be constants, and let $h \to (A, B, C)$ be a field on the three-dimensional torus ([4, 14, p. 72, 15]), that is

$$h = (A\sin z + C\cos y,\ B\sin x + A\cos z,\ C\sin y + B\cos x) \tag{3.5}$$

Then as a solution Euler's equations (3.4), (3.5) we have a field that "drifts" over the three-dimensional torus (A, B, C). From this it follows that (A, B, C) is a field, which does not possess Lyapunov stability as a solution of Euler's equations. Namely, the following is true.

Corollary 3.1. For sufficiently long time t, the ϵ-variation (A, B, C) of the field h in the norm L^2 (and in Sobolev's norm W^n for arbitrary $n \geq 1$) leads to the variation $\sqrt{2}\,\|h\|$ of the corresponding solution of Euler's equations in the norm L^2.

Proof. Let us take $a = b = c = \epsilon/\sqrt{6\pi}$. Then the vector field $\upsilon^t = (a, b, c) + h$ will be the ϵ-variation of h. During the time $T = \pi\sqrt{6\pi}/(2\epsilon)$ the solution of Euler's equations with the initial data υ has the form

$$\upsilon_T = (a, b, c) + (-A\cos(z) + C\sin(y), -B\cos(x) + A\sin(z), -C\cos(y) + B\sin(x))$$

This gives the desired variation of the solution of Euler's equations.

We will now relate the class of solutions of Euler's equations obtained with the groups of the second kind constructed above.

Theorem 3.2. Suppose that, on the two-dimensional compact Riemannian manifold M with $O(2)$-structure, the Killing vector field u is given, and the vector field υ belongs to the Lie algebra r for the generalized group of currents R with pointwise automorphisms of the $O(2)$-structure.

Then, on the spherical bundle $S(M)$ for the pair (u, υ) (here we mean by u its natural lifting to $S(M)$), the conditions of Theorem 3.1 in the version of condition 3′ are satisfied.

Proof. The satisfaction of the condition $\nabla_\upsilon u = 0$ and the stationarity of the vector field υ should be verified. Let us fix the point $x \in M$. Since the spherical bundle π: $S(M) \to M$ is locally trivial, one can choose the neighbourhood $U(x) \subset M$, over which it is constructed as a direct product. Let $V = \pi^{-1}(U)$. We have that $V = U \times S^1$ is a direct product of Riemannian manifolds. We specify a local system of coordinates $p = (p^1, p^2)$ in the vicinity of U and choose on the metrized manifold S^1 the standard coordinate ϕ taken modulo 2π. We have

$$u = \sum_i u_i(p)\partial p^i, \quad \upsilon = \upsilon\partial\phi$$

From this it follows that

$$\nabla_\upsilon u = \sum_j \upsilon_i(p, \phi)u_j(p)\nabla_{\partial\phi}\partial p^j$$

Since the metric of the direct product is given on V, then $\nabla_{\partial\phi}\partial p^j = 0$, and from this it follows that $\nabla_\upsilon u = 0$. Next we calculate $\nabla_\upsilon \upsilon$. For fixed p we denote the vector field on S^1 by $b(p) = \upsilon(p, \phi)\partial\phi$. It is directly verified that $\nabla_\upsilon \upsilon(p, \phi) = \nabla_b b$, where a covariant derivative of the vector field b along itself on S^1 is on the right-hand side. The vector field b is orthogonal for the standard metric on S^1, where it follows that $b(\phi) = \text{const}$, and we obtain that $\nabla_b b = 0$. This means that the vector field υ is stationary and the conditions of Theorem 3.1 are satisfied.

4. THE CASE OF A VISCOUS INCOMPRESSIBLE FLUID

Let us consider the flow of a viscous incompressible fluid described by the Navier–Stokes equations ([16, 17])

$$\partial\upsilon/\partial t + \nabla_\upsilon \upsilon - \nu\Delta\upsilon = \nabla p \tag{4.1}$$

where Δ is the Laplace–Beltrame operator on vector fields. Here solutions integrable at infinity are also obtained.

Consider the vector fields on a three-dimensional torus from Proposition 3.1. Suppose $u = (a, b, f(x, y))$ on T^3, $a, b \in \mathbf{R}$ is the initial velocity field of the viscous incompressible fluid. The Navier–Stokes equations for the velocity field of the form u lead to the system

$$\partial a/\partial t = 0, \quad \partial b/\partial t = 0, \quad \partial f/\partial t + a\partial f/x + b\partial f/\partial y - \nu\Delta f = 0$$

where Δf is the Laplace operator. For further analysis it will be more convenient to use a Fourier-series expansion of the function f

$$f = \sum_{(k,l)} f_{k,l}(x, y)$$

$$f_{k,l}(x, y) = (a_{(k,l)}\cos(kx + ly) + b_{(k,l)}\sin(kx + ly))$$

We put $\lambda_{(k,l)} = -k^2 - l^2$.

Next we introduce the time-dependent function

$$f_t(x, y) = \sum_{k,l} \exp(\nu\lambda_{(k,l)}t) f_{k,l}(x, y) \tag{4.2}$$

Proposition 4.1. The curve $u^t = (a, b, f_t(x - at, y - bt))$ is a solution of the Navier–Stokes equations on T^3 with the initial conditions $u = (a, b, f(x, y))$, $a, b \in \mathbf{R}$.

Proof. To solve Euler's equations with the same initial data u we use the method of variation of parameters. As the varying parameters we will use the Fourier-expansion coefficients of the function f.

Since $\lambda_{(k,l)}$ is the eigenvalue of the Laplace operator on prime harmonics $(\cos(kx + ly), \sin(kx + ly))$ we obtain equations for finding the Fourier-expansion coefficients

$$\partial a_{(k,l)}/\partial t = \nu\lambda_{(k,l)}a_{(k,l)}, \quad \partial b_{(k,l)}/\partial t = \nu\lambda_{(k,l)}b_{(k,l)}$$

where the validity of the proposition follows.

Thus, for the Navier–Stokes equations, as well as for Euler's equations, solutions on the three-dimensional torus with initial conditions $u \in b$ give the curves $u^t \in b$, extendable to infinity in time. The corresponding flows U^t of viscous incompressible fluid lie in the group B. As a result, the infinite-dimensional Lie group B is an invariant group for evolutionary equations both of an ideal fluid and of a viscous incompressible fluid. We will give an analog of Theorem 3.1 for a viscous fluid.

Theorem 4.1 Suppose that, on the compact oriented Riemannian manifold M we are given two divergence-free vector fields u and υ that obey conditions 1, 2 and 3 (or condition 3′) of Theorem 3.1 and, moreover, are eigenvectors for the Laplace-Beltrame operator with eigenvalues λ and μ respectively.
Then the vector field

$$w^t = \exp(\nu\lambda t)u + \exp(\nu\mu t)g_*^{-\phi(t)}(\upsilon) \text{ under condition 3}$$

or

$$\exp(\nu\lambda t)u + \exp(\nu\mu t)g_*^{\phi(t)}(\upsilon) \text{ under condition 3}'$$

will be a solution of the Navier–Stokes equations for a viscous incompressible fluid with initial data $u + \upsilon$ (where, if $\lambda \neq 0$, then $\phi(t) = (\exp(\nu\lambda t) - 1)/(\nu\lambda)$ when $t \neq 0$ and $\phi(0) = 0$, and, if $\lambda = 0$, then $\phi(t) = t$).

Proof. Here we will also use the method of variation of parameters. To fix our ideas, suppose condition 3 is satisfied. We will seek a solution in the form

$$w^t = a(t)u + b(t)g_*^{-\phi(t)}(\upsilon)$$

If we take into account that, when the Riemannian metric is conservative, the action of diffeomorphisms on vector fields is commutative with the action of the Laplace–Beltrame operator, we obtain equations in a and b.

$$\partial a/\partial t = \nu\lambda a, \quad \partial b/\partial t = \nu\mu b$$

with initial conditions $a(0) = b(0) = 1$, which must be supplemented by an equation in ϕ. Substituting the expression for w^t into the Navier–Stokes equations we obtain

$$\partial t(g_*^{-\phi(t)}(\upsilon))_{t=\tau} = [\exp(\nu\lambda\tau)u, g_*^{-\phi(t)}(\upsilon)]$$

Using the well-known expression for the Lie bracket of vector fields in terms of the derivative with respect to action of the curve of diffeomorphisms from the flux of one vector field to another ([9, p. 101–105]) and taking into account the change of time t that has been made, we obtain the following equation for $\phi(t)$:

$$\partial\phi/\partial t = \exp(\nu\lambda t)$$

with the initial condition $\phi(0) = 0$. Solving the equations obtained for a, b, ϕ we complete the proof of the theorem.

5. THE RELATION BETWEEN THE CONSTRUCTIONS AND FIELD THEORY

We will return to Example 2.2 and analyse the case $n = 2$ in more detail. The group of conformal transformations of tangent space at the point $CON(T_xS^2) \cong CON(R^2)$ is commutative. Hence it follows that the action (2.1) of the group K on $R = CON(S^2)$ reduces to shifts of smooth functions on S^2 by conformal transformations of S^2. Namely, the conformal transformation of the plane $t \in CON(R^2)$ can be specified by the pair of numbers $t = (\phi, \lambda)$, where ϕ is the angle of clockwise rotation of the plane and λ is the value of homothety. From this it follows that the element $t \in CON(S^2)$ can be specified by a pair of smooth functions $(\phi(x), \lambda(x))$ which, at the point $x \in SO(1, 3)$, give these parameters for the action in $T_x(S^2)$. Then the action of the element $k \in SO(1, 3)$ on $CON(S^2)$ has the form

$$k(\phi(x), \lambda(x)) = (\phi(k^{-1}x), \lambda(k^{-1}x))$$

The normal subgroup N of the action non-efficiency of the group $B = SO(1, 3)R$ on $S(S^2)$ involves elements of the form $(0, \lambda(x))$. After factorization with respect to N an element in the group $Q = B/N$ can be specified by the pair $(k, \phi(x))$, and as Q we get the semidirect product $SO(1, 3)C^\infty(S^2)$ of the group $SO(1, 3)$ by the space of smooth functions on the sphere $C^\infty(S^2)$, where conformal transformations act on the functional space by translations. This is a Bondi–Metzner–Sachs (BMS) group, which is well-known in field theory [18].

Thus we have proved the following proposition.

Proposition 5.1. The BMS group is a Lie–Frechet group of the second kind and can be realized as a subgroup of the group of diffeomorphisms

$$\mathrm{Diff}(S(S^2)) \cong \mathrm{Diff}(RP^3)$$

In the BMS group we consider its subgroup $L = SO(3)O(S^2)$, which is a semidirect product of the three-dimensional proper orthogonal group $SO(3)$ and the group $O(S^2)$ of pointwise orthogonal transformations on TS^2. The group L also acts on $S(S^2)$. The Lie algebra 1 of the group L is formed by vector fields of this action that have the form $u = h + r$, where h is the vector field of the action of the orthogonal group on S^2, and r is the vector field of the pointwise $o(2)$-automorphisms of Riemannian metric on S^2. According to Theorem 3.2, such vector fields give solutions of Euler's equations on $S(S^2)$ of the form $h + g^t_*(\upsilon)$, where g^t is the flux of the orthogonal vector field h on S^2. Note that $L \subset B = SO(3)CON(S^2)$ with $N \cap L = \mathrm{Id}$, i.e. the action of the group L on $S(S^2)$ is efficient. Therefore the group L can be considered as $L = SO(3)C^\infty(S^2)$, i.e. as a semidirect product of the proper orthogonal group $SO(3)$ in the space of smooth functions on the two-dimensional sphere $C^\infty(S^2)$, where the orthogonal transformations act on functions by translations.

From this we have the following proposition.

Proposition 5.2. In the BMS group one can separate the subgroup $L \subset \mathrm{BMS}$, $L = SO(3)C^\infty(S^2)$. Vector fields of the Lie algebra 1 of the group L give non-stationary solutions of Euler's equations on the three-dimensional manifold $S(S^2)$ that are extendable to infinity in time and remain in the Lie algebra l. The corresponding flows of an ideal incompressible fluid are represented by curves in the group L, and hence in the BMS group.

As regards the physical interpretation of the objects that arise from bundles over a two-dimensional sphere with a conformal structure and represented by the BMS group see [19].

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